The control of resonance curve using the shape modulation of the scattering region in elastic waveguide

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Fano resonance is a phenomenon that is caused by interactions between continuous and discontinuous states. It was initially investigated in optical and atomic systems. Although flexural waves have seldom been discussed in association with Fano resonance, they have attracted increased interest because a numerical method, RTM (Recursive Transfer Method), has recently been developed under the framework of weak form discretisation. A flexural wave on an elastic waveguide is a typical biharmonic system governed by a fourth-order differential equation and it is a suitable subject for verifying the feature of RTM that can extract the localised/quasi-localised wave. The continuous and discontinuous states can be realised by the incident and localised waves in an elastic-waveguide scattering problem. When an elastic plate is cut from both sides, the resulting thin region works as a scattering region and the localised wave is generated there. Furthermore, RTM predicts that the shape modulation of the thin region will induce a change in the resonance curve. The purpose of this study is to demonstrate the asymmetric change of resonance curve through a scattering experiment with elastic waveguides made of aluminium. For this purpose, we propose an accurate procedure for extracting the $S$-parameters from the standing waves without needing to alter the fixed boundary condition as was unavoidable in the previous procedure. The data obtained by experiment are consistent with numerical results obtained by RTM, and control of the asymmetry in the resonance curve is demonstrated.
1. INTRODUCTION

The elastic properties of plates and films have attracted much attention in various systems such as dosing organic films with impurities\(^1\), the thermal conductance of narrow wires\(^2,3\), and microwave absorption by piezoelectric materials\(^4\). The application of elastic waves has also been investigated in such systems as an air pump made of elastic plates\(^5,6,7\).

Fano resonance is caused by interactions between the continuous and discontinuous states. It was initially investigated in optical and atomic systems. A typical feature of Fano resonance appears in the asymmetry of the resonance peak that is detected in the frequency-dependent reflectance in the scattering problem\(^8\). As a system that is governed by second-order differential equations, acoustic scattering in an air-core has been discussed\(^9,10\). In contrast with this system, the flexural wave on an elastic plate provides a biharmonic system that is governed by a fourth-order differential equation\(^11\).

By cutting an elastic plate from both sides, a thin scattering region is formed upon which localised waves tend to exist. Using a waveguide made of aluminium plate, the resonance reflection caused by Fano effect was verified by experiment and numerical analysis\(^12\). However, the experiment was not always sufficiently accurate due to variation caused by unavoidable alternation of the two boundary conditions\(^13\). The purpose of this study is to improve the experimental accuracy by developing a new procedure to extract the S-parameters from the displacement data of the flexural wave. Control of the asymmetry in the resonance curve by modulating the shape of scattering region will also be experimentally and numerically demonstrated.

To extract the elements of the S-parameter composed of transmission and reflection coefficients, an accurate procedure is needed to separate travelling waves from displacement data monitored from an oscillating plate. If a boundary condition that does not induce reflection can be realised\(^14\), the accuracy of the S-parameter extraction will be improved. However, the perfectly non-reflective end is much harder to realise than a perfect reflective end. Therefore, we propose a new procedure using one fixed boundary condition of perfect reflection, assuming that the scattering region is symmetric under space inversion. Compared to the previous procedure\(^13\), this new technique is accurate because the plate state is no longer influenced by the previously unavoidable alternating of boundary conditions.

A fitting model for interpolating the experimental data is also developed, by which the frequency dependence of the wave amplitude is approximated with a zero-pole pair based on analytical prolongation. The asymmetry of the resonance curve is determined by the lateral difference between the zero and the pole. The change of the resonance curve's asymmetry is consistent with numerical results obtained by RTM (Recursive Transfer Method), which is a newly developed numerical method under framework of the weak-form discretisation for analysing Fano resonance\(^15\).

The paper is organised as follows. In Section 2, the elastic waveguide and the localised wave is illustrated using the dispersion relation of travelling and damping/increasing waves. In Section 3, the configuration of the experimental equipment is explained and a new procedure for defining the resonance curve is developed. In Section 4, the conclusion of this study is stated.
A. ELASTIC WAVEGUIDE AND LOCALISED WAVE

A.WAVEGUIDE WITH A THIN SCATTERING REGION

Figure 1 shows elastic waveguides made of an elastic plate of width $a$ and thickness $b$. In its centre, the plate is thinned by cutting equivalently from both horizontal sides until the thickness is $0.8b$ ($=b'$). The shape of the thin region is (a) rectangular and (b) rhomboid and the length of the thin region along the $y$-axis is $d$. The length of the input/output region adjacent to the thin region is $L_{in}/L_{out}$. The thin region works as a scatterer and the localised/quasi-localised wave tends to appear since the incident wave is multiply reflected in this region. When the frequency of the incident wave is equal to the eigenfrequency of the localised/quasi-localised wave, Fano resonance occurs and the reflectance becomes unity.

Figure 2 shows the scattering wave analysed by the finite element method in which the right end, $y = L_{out}+d$, is fixed and the left end, $y = -L_{in}$, is shaken with frequencies (a) 710.5 Hz and (b) 708.1 Hz. The width and thickness of the waveguide are $a = 85.0$ mm and $b = 1.0$ mm, respectively, and the thickness of the thin region is $b' = 0.80$ mm; these values are common to both panels (a) and (b). The lengths of the input, output and scattering regions are (a) $L_{in} = 217$ mm $L_{out} = 296$ mm, $d = 75.0$ mm; (b) $L_{in} = 218$ mm $L_{out} = 300$ mm $d = 71.0$ mm, respectively. The elastic constants for aluminium are Young's modulus $E = 68.3$ GPa, Poisson's ratio $\nu = 0.339$ and mass density $\rho = 2.70 \times 10^3$ kg/m$^3$.

The driving frequency was set slightly higher than the frequency at which the resonance reflection occurs. A localised wave of large amplitude is generated in the thin scattering region and the damping wave-tails penetrate into the input and output regions. The oscillation along the $y$-axis is caused by the standing wave that is formed by superposition of the incident and reflection waves. Here, the oscillation amplitude in the output region, $y > d$, is smaller than that in the input region, $y < 0$. This amplitude difference is caused by the fact that a large amount of the incident wave is reflected in the scattering region.
B. WAVEGUIDE MODES AND THE LOCALISED WAVE

Figure 3 shows the dispersion relation of the waveguide mode generated in the waveguide with a uniform width $a = 85 \text{mm}$. The black and light-coloured dots indicate the data for thicknesses $b = 1.0 \text{mm}$ and $b = 0.80 \text{mm}$, respectively. The elastic constants are the same as those used in Figure 2. The details of the calculation about the dispersion relation were explained in Reference 15. The asymmetric mode is not shown because only symmetric modes appear in this experimental setting. The vertical axis is the frequency of the modes. The horizontal axis is the wavenumber or the damping/growing rate that is converted to a dimensionless quantity by multiplying the length $d (= 75 \text{mm})$ for comparison with the size of the scattering region.

The right panel includes the branches of the travelling modes labelled $S_{0}$ and $S_{1}$. As the $S_{1}$ mode has the minimum frequency, the $S_{0}$ mode is the only one travelling mode that can be stimulated when the waveguide is shaken with a frequency below the minimum frequency of $S_{1}$. The left panel includes the branches of the damping/growing mode, where those labelled $S_{10}$ and $S_{11}$ ($S_{00}$ and $S_{01}$) are continuously connected to the $S_{1}$ mode ($S_{0}$ mode) at $|kd| = 0$.

The horizontal dashed line in Figure 3 indicates the peak frequency at which the reflectance reaches its peak in the scattering experiment discussed in Section 3. Using this diagram, the oscillation of the waveguide at this peak can be speculated as an almost localised wave composed of the travelling and damping/growing waves. The peak frequency indicated by the dashed line exists between the two minimum frequencies of the $S_{1}$ modes in the input/output region with $b = 1.0 \text{mm}$ and the scattering region with $b = 0.8 \text{mm}$. Therefore, a kind of travelling wave similar to the $S_{1}$ mode can be generated in the scattering region. This wave is multiply reflected in this region and forms an almost-localised wave when the half-wavelength is approximately the length $d$, $kd \approx \pi$. A certain amount of the localised wave is transferred to the adjacent input/output regions and changes not only the damping-wave tails but also a travelling wave. As a result, the wave generated in the scattering region can be regarded as quasi-localised, and dissipates its wave energy by emitting a travelling wave with the $S_{0}$ mode. The wave tails penetrating into the input and output regions can be approximated by superposition of the $S_{0}$, $S_{10}$ and $S_{11}$ modes. The open circles in Figure 3 correspond to these three modes that contribute to forming the quasi-localised wave.
3. EXTRACTION METHOD OF THE S-PARAMETER

A. EXPERIMENTAL SYSTEM

Figure 4 shows the configuration of the experimental equipment. The left end of the waveguide shown in Figure 1 is connected to the shaker (EMIC 512-A), which shakes along the z-axis and emits an incident wave propagating along the y-axis. The right end of the waveguide is fixed by a vice, through which the travelling wave is perfectly reflected and a standing wave appears.

The driving power of the shaker is provided by the amplifier (PIONEER A-D1) using a sinusoidal signal from the generator (NF1930A). The vertical displacement of the waveguide is detected by the displacement sensor (KAMAN KD2306-1SU). The amplitude and phase of the displacement are monitored by the oscilloscope (Tektronix TDS 2004B). The monitoring sites are indicated by the five black circles located in each input and output region.

![Figure 4. Configuration of experimental equipment.](image)

B. EXTRACTION OF RESONANCE CURVE

i. Scattering with standing waves

The symbol $a_1$ and $a_2$ are used to express the amplitudes of the wave that enters the scattering region from the input and output regions, respectively. Furthermore, the symbol $b_1$ and $b_2$ are used for amplitudes of the outgoing waves from the scattering region. Then, we can assume that $b_1 = a_1^*$, $b_2 = a_2^*$ because the waves in input and output regions are standing waves. Here, the amplitudes are expressed using a complex form with the absolute value and phase. The asterisk $^*$ represents the complex conjugate. According to the general formulation of a linear system, these complex amplitudes satisfy the relation as follows,

$$
\begin{bmatrix}
  a_1^* \\
  a_2^*
\end{bmatrix} = S
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix}
$$

(1)

Here, the symbol $S$ is the scattering matrix, whose components are called the $S$-parameter. These components are equivalent to the reflection and transmission coefficients and the matrix $S$ can be expressed as follows,
Here, the symbols $r$ and $t$ are the reflection and transmission coefficients, respectively, when the input comes from the left end of the waveguide. The symbols with the dash, $'$, indicate the case where the input comes from the right side.

In the previous procedure for measuring the displacements of the oscillating plate to determine the $S$-parameters, we needed to alter the boundary condition of the right end during measurements\(^\text{13}\). However, the alternation of the boundary condition induces the accuracy deterioration. Therefore, we improve the measurement procedure by imposing symmetry under space inversion upon the shape of scattering region. In addition to this symmetry, the system was assumed to be symmetric under time reversal and to satisfy energy conservation. These conditions can be expressed as follows,

$$r = r', \quad t = t', \quad r^* t' + r' t^* = 0.$$  \hspace{1cm} (3)

Using conditions (3), the independent unknowns in (1) are reduced to two complex coefficients $r$ and $t$. Then, the determinant of $S$ is expressed as $\det[S] = -t/l^* = r/l^*$. Therefore, we can solve (1) for the coefficients $r$ and $t$ as follows,

$$r = (|a_1|^2 - |a_2|^2)/(a_1^2 - a_2^2),$$  \hspace{1cm} (4)

$$t = (a_1 a_2^* - a_1^* a_2)/(a_1^2 - a_2^2).$$  \hspace{1cm} (5)

Using the absolute ratio $|a_2/a_1| = \sqrt{\alpha}$ and the phases $\theta_1 = \arg[a_1]$ and $\theta_2 = \arg[a_2]$, the reflectance $|r|^2$ can be expressed as follows,

$$|r|^2 = 1 - \frac{4\alpha}{(1 + \alpha)^2 + (1 - \alpha)^2 \cot^2(\theta_2 - \theta_1)}. \hspace{1cm} (6)$$

Ultimately, the quantity for determining the reflectance $|r|^2$ is reduced to only the amplitude ratio $a_2/a_1 = \sqrt{\alpha} \exp[i(\theta_2 - \theta_1)]$.

**ii. Wave amplitude separated from the displacement data**

When the frequency is less than the minimum frequency of the $S_1$ mode, the wave in the input/output region can be approximated by the superposition of three modes, $S_0, S_{10}$ and $S_{11}$, which are indicated by open circles in Figure 3. Among these modes, the $S_0$ mode is related to the amplitude $a_1$ in the input region. Similarly, amplitude $a_2$ is also the amplitude of the $S_0$ mode in the output region.

For extracting the amplitudes $a_1$ or $a_2$ from measured displacements, we need to separate the $S_0$ mode from the displacements. As shown in Figure 4, the displacements are monitored by the oscilloscope from five sites $y = y_n$ ($n = 1, 2, ..., 5$) along the centre line, $x = 0$. The gross span of the five sites is set to almost the half-wavelength to ensure extraction accuracy.

In the input region, $0 < y < L_{in}$, the displacement $u(y_n)$ at $y = y_n$ can be approximated by the superposition of the waveguide modes, $u_{\text{apx}}(y_n)$, defined as follows,

$$u_{\text{apx}}(y_n) = a_1 e^{ikx} + a_1^* e^{-ikx} + c_1 e^{k\gamma_n} + d_1 e^{k\gamma_n}.$$  \hspace{1cm} (7)
Here, the first sum of two terms in the right hand side represents the standing wave that is composed by the $S_0$ mode. The coefficients $c_1$ and $d_1$ are the amplitude of the $S_{10}$ and $S_{11}$ modes, respectively, which compose the damping-wave tail. The symbol $k$ denotes the wave number of the travelling $S_0$ mode. The symbol $k'$ and $k''$ are the damping/growing rate of the $S_{10}$ and $S_{11}$ modes, respectively. The value of $k$, $k'$ and $k''$ are equal to the horizontal coordinates of open circles in Figure 3. The values of $a_1$, $c_1$ and $d_1$ are determined so as to minimise the quantity $Q$ defined by

$$Q = \sum_{n=1}^{5} \left| u(y_n) - u_{apx}(y_n) \right|^2 \quad (8)$$

Similarly, the amplitude $a_2$ in the output region, $y > L_{out} + d$, is also extracted from the displacement data.

**C. FITTING AND ASYMMETRY OF THE RESONANCE CURVE**

**i. Fitting model and asymmetry parameter**

Figure 5 and 6 show the resonance curves that were determined within the scattering regions of the rectangle and the rhombus, respectively. The ratio of wave amplitudes, $a_2/a_1$, is determined from the measured displacements using equation (7). Then, the reflectance $|r|^2$ is derived by (6). Here, panel (a) shows the absolute amplitude, $\sqrt{\alpha} (= |a_2/a_1|)$, panel (b) shows the phase difference, $\theta_2 - \theta_1 (= \arg[a_2/a_1])$, and panel (c) shows the frequency dependence of the reflectance, $|r|^2$, which is called the resonance curve in the narrow sense. The horizontal axis is equivalent to the driving frequency of the shaker. The black dots show the experimental data and the curves are the approximation curves drawn by using a fitting model in which the absolute

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**Figure5. Resonance curves with rectangular scattering region.**

**Figure6. Resonance curves with rhombus scattering region.**
amplitude and the phase difference are independently fitted.

As has already been reported, the reflectance $|r|^2$ becomes unity at the peak and the resonance reflection occurs\(^\text{12}\). The peak frequency $f_{\text{peak}}$ is 709.4 Hz in Figure 5 and 698.1 Hz in Figure 6. The peak of the absolute value in panel (a) and the rapid change of the phase difference $\theta_2 - \theta_1$ in panel (b) coincide with the resonance peak in panel (c). Furthermore, the resonance curve is asymmetric. Therefore, the resonance can be regarded as Fano resonance\(^\text{8}\).

The fitting model is built under the framework of analytical prolongation in frequency space. The square of the amplitude ratio, $(a_2/a_1)^2$, is regarded as a function of the frequency $f$ and assumed to be approximated by a pair comprising the zero $f_{\text{zero}}$ and the pole $f_{\text{pole}}$ as follows,

$$
\frac{(a_2/a_1)^2}{f_{\text{zero}} - f_{\text{pole}}} \approx \frac{f - f_{\text{zero}}}{f - f_{\text{pole}}}.
$$

(9)

Since $(a_2/a_1)^2 = \alpha \exp[i2(\theta_2 - \theta_1)]$, the absolute value and the half phase of (9) are $\alpha$ and $\theta_2 - \theta_1$, respectively. The zero and the pole listed in panel (a) in Figure 5 and 6 are values obtained by manual fitting. The difference between the zero and the pole causes the asymmetry in the frequency dependence of the absolute amplitude, and influences the resonance curve. The frequency dependence of the phase difference is also related to the asymmetry. According to (6), the reflectance, $|r|^2$, becomes unity, when the phase difference satisfies $\cot^2(\theta_2 - \theta_1) = \infty$. Therefore, a peak coincidently occurs in the case $\theta_2 - \theta_1 = 0$. If the frequency at which the phase difference $\theta_2 - \theta_1$ vanishes does not equal to the centre of fitting curve, the resonance curve becomes asymmetric.

In Figure 6, the lateral values of $f_{\text{zero}}$ and $f_{\text{pole}}$ are approximately equal, therefore, the peak frequency $f_{\text{peak}}$ can be regarded as satisfying the relation $f_{\text{peak}} = \text{Re}[f_{\text{zero}}] = \text{Re}[f_{\text{pole}}]$. In this case the fitting curve derived from (9) is symmetric around the peak frequency. If the values of $f_{\text{zero}}$ and $f_{\text{pole}}$ have any lateral discrepancy, the frequency dependences of $\sqrt{\alpha}$ and $\theta_2 - \theta_1$ become asymmetric as shown in Figure 5. Therefore, the lateral difference, $\text{Re}[f_{\text{zero}} - f_{\text{pole}}]$, is a quantity for measuring the asymmetry of the resonance curve. On the contrary, the vertical values $\text{Im}[f_{\text{zero}}]$ and $\text{Im}[f_{\text{pole}}]$ serve as measures to define the width of the resonance curve.

i. Numerical estimation of resonance asymmetry by RTM

Figure 7 shows the resonance curves obtained by the newly developed numerical method, RTM. Details of the calculation procedure are given in References 11 and 15. Panels (a) and (b) show the cases where the rectangle and the rhombus are used as the scattering region,
The asymmetry/symmetry of the resonance curve depends on the phase difference of $a_1$ and $a_2$ as indicated by (6), which is induced by the phase change of the incident wave as it is transmitted through the scattering region and interacts with the quasi-localised wave. As the scattering region of rhombus is held by a couple of two horn-like parts of the input/output regions, the quasi-localised wave simultaneously oscillates with the incident wave and the phase difference tends to disappear. Therefore, the resonance curve in panel (b) is symmetric in comparison with that in panel (a).

The peak frequency $f_{\text{peak}}$ is (a)708.9 Hz and (b) 707.0 Hz. These values differ from the experimental frequencies by (a) 0.6 Hz and (b) 8.9 Hz. Though a certain amount of difference appears in the rhombus data (b), the asymmetry of the resonance curve is consistent between the experiment and the numerical analysis.

4.CONCLUSION

The resonance phenomenon caused by the quasi-localised wave generated in the scattering region was experimentally and numerically investigated.

To address the requirement to improve the extraction accuracy of the $S$-parameters from the displacement data detected from an oscillating plate, we proposed a new experimental procedure for separating the amplitudes of the travelling waves from the standing waves in the input/output region. This procedure was realised to impose space-inversion symmetry upon the shape of scattering region, in addition to the conventional conditions of time-reversal symmetry and energy conservation. We also proposed a new fitting model for the resonance curve using a pair comprising a zero and a pole existing in an analytical prolongation of the frequency space. The lateral difference between the zero and the pole was identified as an asymmetric measure of the resonance curve. We demonstrated the possibility of asymmetric control of the resonance curve by modulating the shape of the scattering region. For this demonstration, a new procedure for extracting the $S$-parameters and the fitting model were effectively used.

As a next step, we plan to accurately verify the numerical RTM method using the procedure developed in this study. RTM is expected to serve various fields such as vibration analysis and ultrasonic diagnosis.

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