Journal of the Physical Society of Japan
Vol. 54, No. 1, January, 1985, pp. 236–244

Nonlinear Waves on an Electron-Charged Surface of a Liquid ^{4}He Film

Hatsuhiro KATO

Department of Engineering Science, Faculty of Engineering, Hokkaido University, Sapporo 060

(Received September 14, 1984)

A study is made of nonlinear waves which appear on the surface of a ^{4}He film when the film loads electrons on its surface. By using the reductive perturbation method it is found that the nonlinear waves obey the Benjamin-Ono equations or the Korteweg-de Vries equation according to the conditions imposed on the system.

§1. Introduction

A remarkable feature of nonlinear waves is a formation of stably propagating solitary waves called solitons which exist because of the dynamical balance between dispersion and nonlinear effects. Solitons are now playing important roles in various fields of physics. In the system of superfluid ^{4}He film the study of nonlinear surface waves is regarded as naturally extended one of the linear third sound. In this viewpoint Huberman and Nakajima *et al.* considered a nonlinear wave in a thin liquid ^{4}He film,[1-3] and predicted a possible ordered train of solitons. Some evidence of such a soliton train has been experimentally observed by Kono *et al.*[4] On the other hand, there are some investigations of the electron-charged surface of liquid ^{4}He. In this system the existence of electrons involves the softening of the surface wave indicating that the group velocity vanishes at particular wave numbers. Moreover the softening can be controlled by an external electric field applied perpendicularly to the helium surface. Taking into account of this property, Mima and Ikezi[5] pointed out a possibility of a surface envelope soliton which obeys the nonlinear Schrödinger equation. The above-mentioned consideration was performed under the condition that the thickness of the ^{4}He film is much larger than the wave length. In this paper we investigate other types of solitons which will be expected when the thickness of the film is small compared with the wave length.

Geometrical configuration of the system is shown in Fig. 1. A liquid ^{4}He film with a thickness d on an appropriate substrate loads electrons on its surface. We take the x axis parallel to the equilibrium surface and the y axis perpendicular to the surface, and consider one-dimensional waves propagating along the x axis. By the external electric field and the image charges which appear both the substrate and the electrodes set at $y=l_1$, $-l_2$, the electrons are pushed against the surface. We introduce pressure p_e to describe this effect. The external field can be controlled by the voltage difference V between the two electrodes. We indicate the displacement of the surface as $a(x, t)$ and the dielectric constants of a ^{4}He vapor, liquid ^{4}He and the substrate as ε_1, ε_2 and ε_3 respectively. The maximum electron density which the surface can support is about 10^9 cm^{-2}.[6] In this paper the electrons

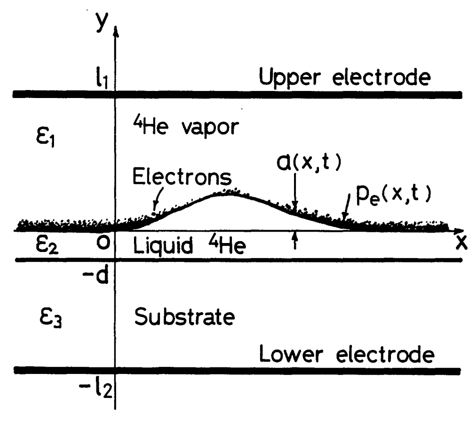

Fig. 1. Geometrical configuration of the system.

are treated as a continuously charged object so that the wave length must be much larger than the interval between the electrons. If we consider then the wave length is not small compared with the film thickness d, the relation $d \gg \sqrt{10^{-9}}$ cm should be satisfied.

Utilizing hydrodynamics and electrostatics, the equations are derived which govern the waves on the charged surface of the liquid ^{4}He film. These equations are so complicated to be treated rigorously that we use the reductive perturbation method to obtain more simpler equations. The nonlinear equations thus obtained are the Benjamin-Ono (B-O) equations or the Korteweg-de Vries (KdV) equation. When we employ a metal for the substrate we obtain the B-O equation or the KdV equation corresponding to the values of the external electric field. If we use an insulator for the substrate we obtain the B-O equation. These equations have soliton solutions. To see the evolution of the nonlinear waves we estimate the evolution time and the evolution length for which an initial pulse travels until it grows into a soliton. The evolution time is determined from the linearized one of the nonlinear equation. And the evolution length is estimated as the length for which a pulse travels during the evolution time with the velocity of the linear wave.

In §2, the basic equations governing the surface motion of the film are derived with assumptions that liquid ^{4}He is an ideal fluid and the electrons move freely along the surface. A dispersion relation of the surface wave is also derived. In §3, using the reductive perturbation method, we derive the nonlinear equations from the basic equations under three conditions imposed on the system. In §4, considering the evolution time and evolution length, we discuss roughly the evolution of the nonlinear waves. In this paper use is made of a special operator P, which is introduced already by Mima and Ikezi.[5] In the appendix, we shall examine the reductive perturbation method when equations contain the operator P.

§2. Basic Equations

Assuming that liquid ^{4}He is cooled below the λ-point, we treat liquid ^{4}He as an incompressible ideal fluid. Since the super flow is irrotational, we can introduce the velocity potential $\Phi(x, y, t)$ which satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi=0. \tag{2.1}$$

This equation is subject to the kinematical and the dynamical conditions. The kinematical conditions are that no fluid pass through the bottom:

$$\frac{\partial}{\partial y}\Phi(x, -d, t)=0, \tag{2.2}$$

and the continuity equation holds at the surface:

$$\frac{\partial a}{\partial t}+\left(\frac{\partial \Phi}{\partial x}\right)_1\frac{\partial a}{\partial x}-\left(\frac{\partial \Phi}{\partial y}\right)_1=0. \tag{2.3}$$

The dynamical condition requires Bernoulli's theorem at the surface:

$$\left(\frac{\partial \Phi}{\partial t}\right)_1+\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)_1^2+\left(\frac{\partial \Phi}{\partial y}\right)_1^2\right]+ga-\frac{\alpha}{\rho(d+a)^3}-\frac{\tau}{\rho}\frac{\partial^2 a}{\partial x^2}+\frac{p_e}{\rho}=0. \tag{2.4}$$

Here the terms with the subscript 1 are evaluated on the surface, g is the gravity constant, ρ the density of the liquid, ν the surface tension and α the van der Waals constant indicating the strength of the van der Waals force between ^{4}He and the substrate. In this paper we assume that the film is relatively thick so that the effect of the van der Waals force is negligibly small. The effect, however, shall be maintained in order to compare our result with that of Nakajima *et al.*[3] All the effects of electrons loaded on the ^{4}He surface are expressed by the pressure p_e in eq. (2.4). The pressure p_e can be described in terms of the surface displacement $a(x, t)$. Derivation of this relation will be presented in the latter half of this section.

We shall obtain an adequate solution of eq. (2.1) by using an expansion of the form

$$\Phi(x, y, t)=\sum_{n=0}^{\infty}(y+d)^n\phi_n(x, t). \tag{2.5}$$

Substituting this expression into eq. (2.1) and then taking into account of eq. (2.2), we obtain

$$\Phi(x, y, t)=\cos\left[(y+d)\frac{\partial}{\partial x}\right]\phi_0(x, t). \quad (2.6)$$

Substitution of eq. (2.6) into eqs. (2.3) and (2.4) yields the simultaneous equations for $a(x, t)$ and $\phi_0(x, t)$, which express the motion of the surface.

$$\dot{a}+a'u_1-v_1=0, \quad (2.7a)$$

$$\dot{u}_1+u_1u_1'+v_1v_1'+ga'+\frac{3\alpha}{\rho}\frac{a'}{(d+a)^4}-\frac{\tau}{\rho}a'''+\frac{p_e'}{\rho}=0, \quad (2.7b)$$

where $u=\partial\Phi/\partial x$ and $v=\partial\Phi/\partial y$. The dash and the dot indicate derivatives with respect to x and t respectively. Equation (2.7b) is the result of differentiation of eq. (2.4) with respect to x. To proceed further we assume that the maximum amplitude m of $a(x, t)$ is small compared with the film thickness d, while the dominant wave length l is large compared with d. Then we can rewrite eqs. (2.7a) and (2.7b) in terms of two small parameters $\varepsilon=m/d$ and $\delta=d/l$.[7] To the first nonvanishing order in ε and δ, we obtain the following equations.

$$\dot{a}+(d+a)f'+fa'-\frac{1}{6}d^3f'''=0, \quad (2.8a)$$

$$\dot{f}+ff'+\left(g+\frac{3\alpha}{\rho d^4}-\frac{12\alpha}{\rho d^5}a\right)a'-\frac{d^2}{2}\dot{f}''-\frac{\tau}{\rho}a'''+\frac{p_e'}{\rho}=0, \quad (2.8b)$$

where $f(x, t)=\partial\phi_0(x, t)/\partial x$.

Now we derive the relation between $p_e(x, t)$ and $a(x, t)$ under the assumption that the electrons move freely along the surface so that the surface is always equipotential.[5] First, we consider the electrostatic potential $\phi(x, y, t)$ which satisfies

$$\varepsilon\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi=-4\pi\sigma\delta(y-a), \quad (2.9)$$

where $\sigma(x, t)$ is the charge density projected on the x axis, and ε stands for the dielectric constants of the helium vapor (ε_1), liquid helium (ε_2) and the substrate (ε_3). Equation (2.9) is subject to the following boundary conditions.

$$\lim_{y\to\pm a}\ \phi=0, \quad (2.10)$$

$$\frac{\partial\phi}{\partial x}=0 \quad \text{at } y=l_1,\ -l_2, \quad (2.11)$$

and

$$\phi|_{y=-d+0}=\phi|_{y=-d-0}, \quad (2.12a)$$

$$\frac{\partial\phi}{\partial x}|_{y=-d+0}=\frac{\partial\phi}{\partial x}|_{y=-d-0}, \quad (2.12b)$$

$$\varepsilon_2\frac{\partial\phi}{\partial y}|_{y=-d+0}=\varepsilon_3\frac{\partial\phi}{\partial y}|_{y=-d-0}. \quad (2.12c)$$

It is convenient to introduce an operator P which is defined as[5]

$$\mathrm{P}\sum_k F_k\,e^{ikx}=\sum_k |k|F_k\,e^{ikx}. \quad (2.13)$$

Then the solution of eq. (2.9) satisfying the boundary conditions (2.11), (2.12a), (2.12b) and (2.12c) is given by

$$\phi=\begin{cases}-E_+y+[\mathrm{ch}y\mathrm{P}-\mathrm{sh}y\mathrm{P}\cdot\mathrm{L}_+]h_+, & (a<y<l_1)\\ -E_-y+[\mathrm{ch}y\mathrm{P}+\mathrm{sh}y\mathrm{P}\cdot\mathrm{L}_-]h_-, & (-d<y<a)\\ G+E_-[-d+(y+d)/\varepsilon] & \\ +\dfrac{\mathrm{sh}(l_2+y)\mathrm{P}}{\varepsilon\,\mathrm{sh}d\mathrm{P}\,\mathrm{ch}(l_2-d)\mathrm{P}+\mathrm{ch}d\mathrm{P}\,\mathrm{sh}(l_2-d)\mathrm{P}}(h_--G), & (-l_2<y<-d),\end{cases} \quad (2.14)$$

where

$$\mathrm{L}_+ = \coth l_1 \mathrm{P}, \tag{2.15a}$$

$$\mathrm{L}_- = \frac{\varepsilon \mathrm{ch} d\mathrm{P} \mathrm{ch}(l_2-d)\mathrm{P} + \mathrm{sh} d\mathrm{P} \mathrm{sh}(l_2-d)\mathrm{P}}{\varepsilon \mathrm{sh} d\mathrm{P} \mathrm{ch}(l_2-d)\mathrm{P} + \mathrm{ch} d\mathrm{P} \mathrm{sh}(l_2-d)\mathrm{P}}, \tag{2.15b}$$

$$G = \frac{1}{A}\int \mathrm{d}x h_-, \quad \varepsilon = \varepsilon_3/\varepsilon_2.$$

Here, E_+ and E_- are the average electric fields above and below the ^{4}He surface, A is the system length. h_+ and h_- are appropriate functions of x and t, whose explicit forms are obtained by substitution of eq. (2.14) into eq. (2.10). The resulting expressions to the third order in $a(x, t)$ are as follows.

$$h_\pm = E_\pm \left[a \pm \mathrm{PL}_\pm a + \mathrm{PL}_\pm (a\mathrm{PL}_\pm a) - \frac{1}{2} a^2 \mathrm{P}^2 a \right]. \tag{2.16}$$

The average electric fields E_+ and E_- are connected with the charge density σ_0 on the equilibrium surface and the voltage difference V between the upper and the lower electrodes as follows.

$$4\pi\sigma_0 = \varepsilon_1 E_+ - \varepsilon_2 E_-, \tag{2.17a}$$

$$V = -E_+ l_1 - E_- \left(d + \frac{l_2 - d}{\varepsilon} \right) + \frac{1}{A}\int \mathrm{d}x (h_+ - h_-). \tag{2.17b}$$

Equation (2.17a) is obtained by integrating eq. (2.9) in the region which contains the ^{4}He surface, and eq. (2.17b) is yielded by substitution of eq. (2.14) into $V = \phi(y = l_1) - \phi(y = -l_2)$.

To express the pressure p_e due to the electrons in terms of the displacement $a(x, t)$, we write the electrostatic energy as a functional of $a(x, t)$. Substituting eq. (2.14) into the definition of the energy $U = \int \mathrm{d}v \varepsilon (\nabla\phi)^2/8\pi$ and transforming the volume integral to the surface, we obtain

$$U[a] = \frac{1}{8\pi}\int \mathrm{d}x \left[\left(\varepsilon_1 \phi \frac{\partial\phi}{\partial y} \right)_{y=l_1} - \left(\varepsilon_2 \phi \frac{\partial\phi}{\partial y} \right)_{y=-l_2} \right]$$
$$= \frac{1}{8\pi}\int \mathrm{d}x [-\varepsilon_1 E_+ h_+ + \varepsilon_2 E_- h_-] + \mathrm{const.} \tag{2.18}$$

Let the surface of ^{4}He be displaced virtually by the amount δa, then the energy conservation requires

$$\int \mathrm{d}x p_e \delta a = \frac{1}{8\pi}\int \mathrm{d}x \left[-\varepsilon_1 E_+ \frac{\delta h_+}{\delta a} + \varepsilon_2 E_- \frac{\delta h_-}{\delta a} \right] \delta a. \tag{2.19}$$

Here δa is an arbitrary function so that we have the relation

$$p_e = -\frac{\varepsilon_1 E_+^2}{8\pi} S_+ + \frac{\varepsilon_2 E_-^2}{8\pi} S_-, \tag{2.20}$$

where

$$S_\pm = 1 \pm 2\mathrm{PL}_\pm a + 2\mathrm{PL}_\pm (a\mathrm{PL}_\pm a) + (\mathrm{PL}_\pm a)^2 - a\mathrm{P}^2 a - \frac{1}{2}\mathrm{P}^2 a^2.$$

The relation between $\sigma(x, t)$ and $a(x, t)$ is obtained by integrating eq. (2.9) within a small space which contains the ^{4}He surface as follows.

$$\sigma = \frac{\varepsilon_1 E_+}{4\pi} Q_+ - \frac{\varepsilon_2 E_-}{4\pi} Q_-, \tag{2.21}$$

where

$$Q_\pm = 1 \pm \mathrm{PL}_\pm a + \mathrm{PL}_\pm (a\mathrm{PL}_\pm a) - \frac{1}{2}\mathrm{P}^2 a^2.$$

Linearizing eqs. (2.7a) and (2.7b) with respect to $a(x, t)$, we can obtain the dispersion relation as follows.

$$\omega_k^2=\left[g+\frac{3\alpha}{\rho d^4}+\frac{\tau}{\rho}k^2-\frac{1}{4\pi\rho}(\varepsilon_1 E_+^2\tilde{L}_+ +\varepsilon_2 E_-^2\tilde{L}_-)k\right]k\tanh dk, \tag{2.22}$$

where $\tilde{L}_+$ and $\tilde{L}_-$ are respectively the expressions of L_+ (2.15a) and L_- (2.15b) in which the operator P is replaced by $|k|$. From eq. (2.22) we can derive various dispersion relations as ever reported.[5,8,9]

§3. Nonlinear Waves

In this section the nonlinear waves are considered under the three conditions imposed on the system. We apply the reductive perturbation method to eqs. (2.8a) and (2.8b) in order to see the asymptotic behaviors of the surface waves. For the sake of simplicity, we assume henceforth that l_1 and l_2 are sufficiently large compared with the film thickness d so as to satisfy the relations $L_+=1$ and $L_-=\coth(dP+\theta)$ $(\theta=\coth^{-1}\varepsilon_3/\varepsilon_2)$ in eqs. (2.15a) and (2.15b).

Case 1. We first consider a case when a metal is employed for the substrate and the average field $E_\pm$ is different from zero. Since ε_3, the dielectric constant of the substrate, is infinite, we have $L_-=\coth dP$. The same procedure as we derived eq. (2.8b) from eq. (2.7b) yields the following equation from eq. (2.20).

$$p_e'=-\frac{\varepsilon_1 E_+^2}{4\pi}Pa'+\frac{\varepsilon_2 E_-^2}{12\pi}da'''+\frac{\varepsilon_2 E_-^2}{4\pi}\frac{3a-d}{d^2}a'. \tag{3.1}$$

Making use of eq. (3.1), we write eqs. (2.8a) and (2.8b) in the matrix form as follows.

$$\frac{\partial U}{\partial t}+\begin{bmatrix} f & d+a\\ C & f\end{bmatrix}\frac{\partial U}{\partial x}+\begin{bmatrix} 0 & 0\\ -\varepsilon_1 E_+^2/4 & 0\end{bmatrix}P\frac{\partial U}{\partial X}+\left\{\begin{bmatrix} 0 & 0\\ 0 & -d^2/2\end{bmatrix}\frac{\partial}{\partial t}+\begin{bmatrix} 0 & -d^3/6\\ D & 0\end{bmatrix}\frac{\partial}{\partial x}\right\}\frac{\partial^2 U}{\partial x^2}=0, \tag{3.2}$$

where

$$U=(a,f)^t,$$

$$C=g+3\alpha(d-4a)/\rho d^5+\varepsilon_2 E_-^2(3a-d)/4\pi\rho d^2,$$

$$D=-(\tau-\varepsilon_2 E_-^2 d/12\pi)/\rho.$$

The dominant dispersive term comes from the third of the left hand side of eq. (3.2) and the fourth term contributes only to the small correction to the dispersion. We take the first three terms of eq. (3.2) and apply the reductive perturbation given in the appendix to this equation. Then we obtain the following equation.

$$\frac{\partial a}{\partial\eta}+\frac{3}{2c_0}\left(g-\frac{\alpha}{\rho d^4}\right)a\frac{\partial a}{\partial\xi}-\frac{\varepsilon_1 E_+^2 d}{8\pi\rho c_0}\frac{\partial}{\partial\xi}P_\xi a=0, \tag{3.3}$$

$$\xi=x-c_0 t,\quad \eta=t,$$

$$c_0=[d(g+3\alpha/\rho d^4-\varepsilon_2 E_-^2/4\pi\rho d)]^{1/2}. \tag{3.4}$$

This equation is known as the Benjamin-Ono (B-O) equation.[10] It is more familiar to express P_ξ with H as follows.

$$P_\xi a(\xi)=-\frac{\partial}{\partial\xi}Ha(\xi), \tag{3.5}$$

where H represents the Hilbert transformation

$$Ha(\xi)=\frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}d\eta\frac{a(\eta)}{\eta-\xi}. \tag{3.6}$$

Here p.v. stands for the principle value of the integration.

Case 2. We consider a case where a metal is employed for the substrate and E_+ is equal to zero. In this case as the dispersive term in eq. (3.3) vanishes, we have to consider the first four terms of eq. (3.2). The same treatment as in case 1 yields the following equation.

$$\frac{\partial a}{\partial \eta}+\frac{3}{2c_0}\left(g-\frac{\alpha}{\rho d^4}\right)a\frac{\partial a}{\partial \xi}-\frac{d^3}{6c_0}\left(\frac{3\tau}{\rho d^2}-\frac{3\alpha}{\rho d^4}-g\right)\frac{\partial^3 a}{\partial \xi^3}=0, \tag{3.7}$$

$$\xi=x-c_0t,\quad \eta=t,$$

$$c_0=[d(g+3\alpha/\rho d^4-\varepsilon_2E_-^2/4\pi\rho d)]^{1/2}, \tag{3.8}$$

which is known as the KdV equation and agrees with that given by Nakajima *et al.*[3)]

Case 3. An insulator is used for the substrate. In this case ε_3, the dielectric constant of the substrate, is finite. The following B-O equation is derived by the same procedure as in case 1.

$$\frac{\partial a}{\partial \eta}+\frac{3}{2c_0}\left(g-\frac{\alpha}{\rho d^4}\right)a\frac{\partial a}{\partial \xi}-\frac{\varepsilon_1E_+^2+\varepsilon\varepsilon_2E_-^2}{8\pi\rho c_0}\frac{\partial}{\partial \xi}\mathrm{P}_\xi a=0, \tag{3.9}$$

$$\xi=x-c_0t,\quad \eta=t,$$

$$c_0=[d(g+3\alpha/\rho d^4)]^{1/2}. \tag{3.10}$$

In this case, c_0, the velocity of the linear wave at long wave length, is not affected by the existence of the electrons in contrast with eqs. (3.4) and (3.8).

Now we have two kinds of nonlinear equations under the three conditions imposed on the system. The standard forms of these equations are written in the following way,

$$\frac{\partial u}{\partial \tau}+2u\frac{\partial u}{\partial \chi}+H\frac{\partial^2 u}{\partial \chi^2}=0, \tag{3.11}$$

for the B-O equation and

$$\frac{\partial u}{\partial \tau}-6u\frac{\partial u}{\partial \chi}+\frac{\partial^3 u}{\partial \chi^3}=0, \tag{3.12}$$

for the KdV equation. In deriving eq. (3.11) from eqs. (3.3) and (3.9), we transform the variables t, ξ and a into $\tau=t/t_s$, $\chi=\xi/l_s$ and $u=a/l_s$, where t_s and l_s are the units of time and length respectively. We take that $t_s=4c_0/3(g-\alpha/d^4)$ and $l_s=[t_s\varepsilon_1E_+^2d/8\pi\rho c_0]^{1/2}$ for eq. (3.3), and $t_s=4c_0/3(g-\alpha/\rho d^4)$ and $l_s=[t_s(\varepsilon_1E_+^2+\varepsilon\varepsilon_2E_-^2)d/8\pi\rho c_0]^{1/2}$ for eq. (3.9). To obtain eq. (3.12) from eq. (3.7), we use the transformation; $t=-t_s\nu$, $\xi=l_s\chi$ and $a=l_su$, where we take $t_s=4c_0/(g-\alpha/\rho d^4)$ and $l_s=[t_sd^3(3\nu/\rho d^2-3\alpha/\rho d^4-g)/6c_0]^{1/3}$. It is well known that the B-O equation (3.11) has a single soliton solution of the form

$$u=\frac{2V}{V^2(\chi-\tau V)^2+1}, \tag{3.13}$$

and the KdV equation (3.12) has a soliton solution of the form

$$u=-2\kappa^2\,\mathrm{sech}^2(\kappa\chi-4\kappa^3\tau). \tag{3.14}$$

Here V and κ are positive constants.

§4. Discussion

We have considered the nonlinear waves on the electron-charged surface of liquid ^{4}He and obtained the B-O equations and the KdV equation corresponding to the three conditions imposed on the system. These equations have the soliton solutions of the forms (3.13) and (3.14). We would see the evolution of the nonlinear waves by the inverse scattering methods.[7,11)] For simplicity we take, however, a convenient way to define the evolution time from the linearized ones of eqs. (3.11) and (3.12) and to estimate the evolution length with which our solitons may be experimentally concerned.

First we consider the KdV equation (3.12). At $\tau=0$ if we generate an initial pulse with width W of the form

$$u(\chi,0)=\begin{cases}U_0, & |\chi|\leqq W/2,\\ 0, & |\chi|>W/2,\end{cases} \tag{4.1}$$

the pulse evolves according to the linearized one of eq. (3.12) as follows.

$$u(\chi,\tau)=\int_{c_-}^{c_+}U_0\mathrm{Ai}(x)\,\mathrm{d}x, \tag{4.2}$$

where $\mathrm{Ai}(x)$ is the Airy function and $c_\pm=(2\chi\pm W)/(24\tau)^{1/3}$. Solitons exist under the balance between pulse sharpening due to the nonlinear effects and pulse spreading due to dispersion. If we take only the dispersion effects into account, we may consider that the pulse disperses monotonically. We define the evolution time t_e as the time for which the pulse (4.2) travels until its height reduces to 1/10 of its initial value. The time t_e may be considered as the time which it takes for an initial pulse with width W to grow into soliton. From eq. (4.2) we obtain the evolution time approximately

$$t_e = 15\, W^3. \quad (4.3)$$

The same discussion yields the evolution time for the B-O equation (3.11) as follows.

$$t_e = 4\, W^2, \quad (4.4)$$

where we use the form (4.1) as an initial pulse. The evolution length, for which the initial pulse travels until it grows into soliton, may be estimated as $l_e = c_0 t_e$, where c_0 is velocity of the linear wave.

Table I. Estimates of the evolution time t_e and the evolution length l_e. Here d is the film thickness, n the electron density on the surface and c_0 the velocity of the linear wave.

	Case 1	Case 2	Case 3
d (cm)	10^{-2}	10^{-2}	10^{-2}
n (cm^{-2})	10^9	5×10^8	10^9
c_0 (cm/sec)	2.19	2.19	3.13
t_e (sec)	0.046	5.9	0.50
l_e (cm)	0.10	13	1.6

In Table I we calculate c_0, l_e and t_e under the three conditions treated in the previous section. As seen in Table I we set $d=10^{-2}$ cm and $n=10^9$ cm^{-2} except for Case 2. The average fields E_+ and E_- are taken such that $E_+=-E_-=2\pi\sigma_0$ for cases 1 and 3 and $E_+=0$ and $E_-=$

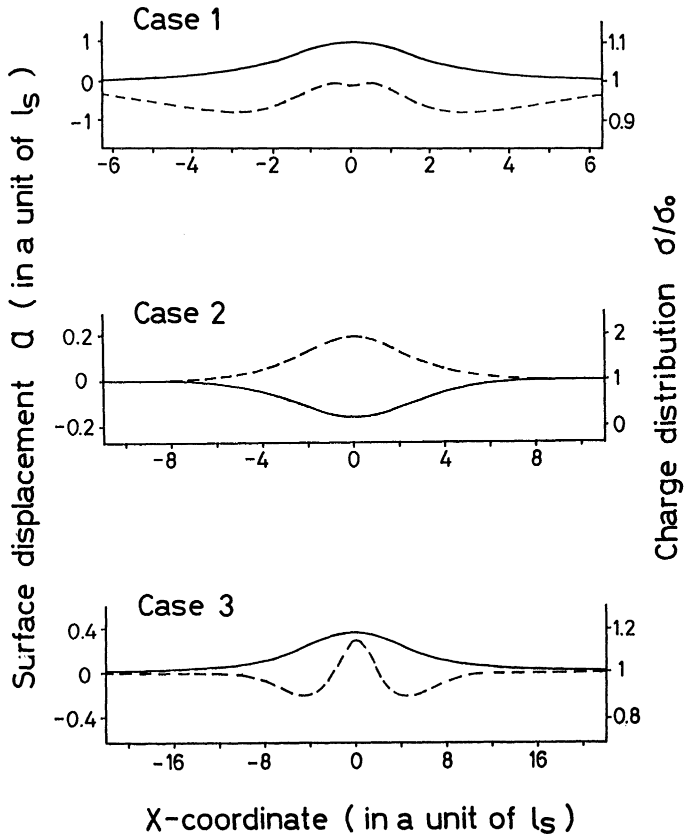

Fig. 2. The surface displacement a (full line) and the charge distribution σ/σ_0 (dashed line) when a single soliton is generated for each case. For each figure the axis of abscissas is located on the bottom of the ^{4}He film.

$-4\pi\sigma_0$ for case 2. For case 3, CaF_2 ($\varepsilon_3=6.7$) is used for the substrate. When we fix the electron density and the voltage between the electrodes, it is important to consider the instability of the surface.[9,12-14] If we take $n=10^9\ \mathrm{cm}^{-2}$ in case 2, the instability will appear at a long wave length and the calculation loses its validity so that we set $n=5\times 10^8\ \mathrm{cm}^{-2}$ for this case. We take W, the width of the initial pulse, to the width of the single soliton whose height is 6/10 of the film thickness. In Fig. 2 such single solitons are depicted. The full line indicates the displacement of the surface and the dashed line does the charge distribution. It should be noted that the charge distribution of the surface satisfies $\sigma/\sigma_0>0$. Since there is no other quantity than the electrons on the surface, any appearance of positive charge destroys the meaning of the present discussion.

If we can solve the problems how to generate an initial-pulse and detect the displacement of the surface, we could catch the existence of solitons.

Acknowledgements

The author would like to thank Professor N. Tokuda for valuable discussions and a critical reading of the manuscript. The author also wishes to thank Professor T. Sakuma and T. Akimoto for useful discussions and Professor H. Ono for his comments on the Benjamin-Ono equation.

Appendix A: The Reductive Perturbation Method Including Operator P

A general discussion on the reductive perturbation method is made by Taniuti and Wei.[15] Since we use in this paper the operator P defined in eq. (2.13), we examine how to modulate their discussion when an equation contains the operator P as follows.

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+\prod_{\alpha=1}^{p}\left(H_\alpha\frac{\partial}{\partial t}+K_\alpha\frac{\partial}{\partial x}+M_\alpha \mathrm{P}_x\right)U=0, \qquad (\mathrm{A}\cdot 1)$$

where U is a colum vector with N components, H_α, K_α, M_α ($\alpha=1,\cdots,p$) and A are $N\times N$ matrices and the elements of them are functions of U. In the third term of the left hand side the operator P_x is included. Here the subscript x indicates the independent variable. A parallel discussion to that of Taniuti and Wei[15] leads to the nonlinear equation of the form

$$\frac{\partial u_n}{\partial \eta}+\frac{c}{r_{0n}}(u_n-u_{0n})\frac{\partial u_n}{\partial \xi}+\mathrm{D}u_n=0, \qquad (\mathrm{A}\cdot 2)$$

$$\xi=x-c_0t,\quad \eta=t,$$

$$c=\frac{1}{L_0R_0}L_0[(R_0\cdot\nabla_{\mathrm{U}})A]_{\mathrm{U}=\mathrm{U}_0}R_0,$$

$$\mathrm{D}=\frac{1}{L_0R_0}L_0\prod_{\alpha=1}^{p}\left[(-c_0H_{\alpha0}+K_{\alpha0})\frac{\partial}{\partial\xi}+M_{\alpha0}\mathrm{P}_\xi\right]R_0.$$

Here U_0 is a nonperturbed constant solution of eq. (A·1), c_0 is an eigenvalue of A_0 ($=A(U_0)$), R_0 and L_0 are the right and left eigenvectors of A_0 with an eigenvalue c_0, u_n, u_{0n} and r_{0n} are respectively n'th components of U, U_0 and R_0 and we put $H_{\alpha0}=H_\alpha(U_0)$, $K_{\alpha0}=K_\alpha(U_0)$ and $M_{\alpha0}=M_\alpha(U_0)$ ($\alpha=1,\cdots,p$).

References

1) B. A. Huberman: Phys. Rev. Lett. **41** (1978) 1389.
2) S. Nakajima, S. Kurihara and K. Tohdoh: J. Low Temp. Phys. **39** (1980) 465.
3) S. Nakajima, K. Tohdoh and S. Kurihara: J. Low Temp. Phys. **41** (1980) 521.
4) K. Kono, S. Kobayashi and W. Sasaki: J. Phys. Soc. Jpn. **50** (1981) 721.
5) K. Mima and H. Ikezi: Phys. Rev. B **17** (1978) 3567.
6) R. S. Crandall: Surf. Sci. **58** (1976) 266.
7) G. L. Lamb: *Elements of Soliton Theory* (John Wiley & Sons, New York, 1980) p. 171.
8) L. P. Gor'kov and D. M. Chernikova: Sov. Phys.-JETP Lett. **18** (1973) 68.
9) K. Mima, H. Ikezi and A. Hasegawa: Phys. Rev. B **14** (1976) 3953.
10) H. Ono: J. Phys. Soc. Jpn. **39** (1975) 1082.
11) A. S. Fokas and M. J. Ablowitz: Stud. in Appl. Math. **68** (1983) 1.

12) P. Leiderer, W. Ebner and V. B. Shikin: Surf. Sci. **113** (1982) 405.
13) R. W. Giannetta and H. Ikezi: Surf. Sci. **113** (1982) 412.
14) P. Leiderer: Phys. Rev. B **20** (1979) 4511.
15) T. Taniuti and C. Wei: J. Phys. Soc. Jpn. **24** (1968) 941.