Weak-Form Discretization Scheme for Recursive Transfer Method

Hatsuhiro Kato[1),] Hatsuyoshi Kato[2)] , Takaaki Ishii[1)]

1） University of Yamanashi
2） NIT Tomakomai College

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Introduction

- Numerical tools for resonance phenomena.

  ⇒ RTM（Recursive Transfer Method）

  for scattering Problems

- Extraction of localized/quasi-localized waves

  ⇒ suitable to Fano resonance

- RTM scope was limitted to 2nd-order differential eq
- Need to derive 2nd order diffrence eq from higher-order differential eqs.

  ⇒ Weak-Form Discritization

Scattering problems（1D）

■ 4th-order diff. eq.

$$[l(x)u''(x)]'' - m(x)u(x) = 0$$

Euler-Bernoulli model for elastic beams

$l(x) = EI$ Stiffness of a beam

$m(x) = \omega^2 \lambda(x)$

$\lambda(x)$ Linear mass density

ω Angular freq.

■ Functional

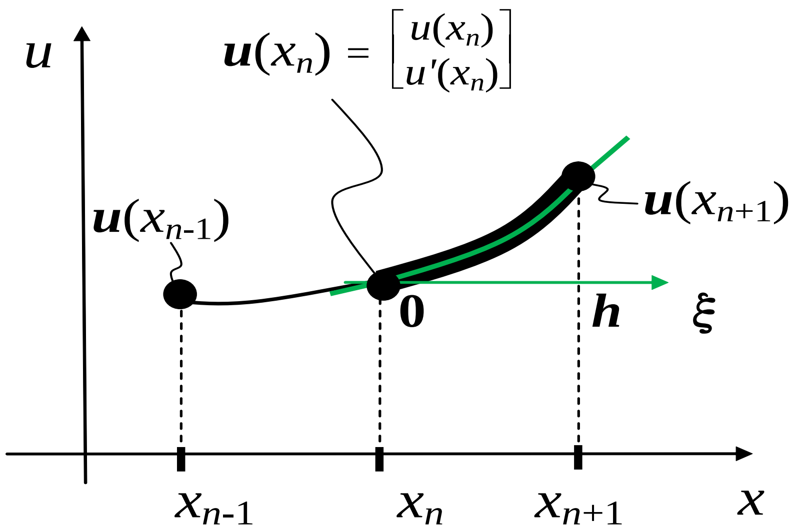
$$F[w,u] = \int_L \{w'' \, l \, u'' + wmu\} dx + \left[-w'(l\,u)'' + w(l\,u)''' \right]_{xn-h}^{xn+h} \Rightarrow 0 \quad (\forall w)$$

Narrow interval $L = [x_n - h, x_n + h]$

BC for $w(x)$

$$w(x_n+h) = 0, \quad w'(x_n+h) = 0$$

Discretization under weak-form scheme



- **Interpolation with** Helmite elemtn

Interpolation coordinate $\xi = x - x_n$

$$u(x) = [1 \quad \xi \quad \xi^2 \quad \xi^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h & h^2 & h^3 \\ 0 & 1 & 2h^2 & 3h^3 \end{bmatrix}^{1} \begin{bmatrix} \mathbf{u}(x_n) \\ \mathbf{u}(x_{n+1}) \end{bmatrix}$$

$$\mathbf{u}(x_n) = \begin{bmatrix} u(x_n) \\ u'(x_n) \end{bmatrix}$$

- **Functional expression**

$$F[w,u] = \int_L \{w'' D\, u'' + vwu\} dx$$

$$\Rightarrow 0 \quad (\forall w) \qquad L = [x_n - h,\ x_n + h]$$

$$c_n \mathbf{u}(x_{n-1}) + b_n \mathbf{u}(x_n) + a_n \mathbf{u}(x_{n+1}) = 0$$

Stepping matrices for port regions

Wave Constants in Ports

$$c_{port}\boldsymbol{u}(x_{n-1}) + b_{port}\boldsymbol{u}(x_n) + a_{port}\boldsymbol{u}(x_{n+1}) = 0$$

$$\boldsymbol{u}(x_{n+1}) = e^{\eta h}\boldsymbol{u}(x_n)$$

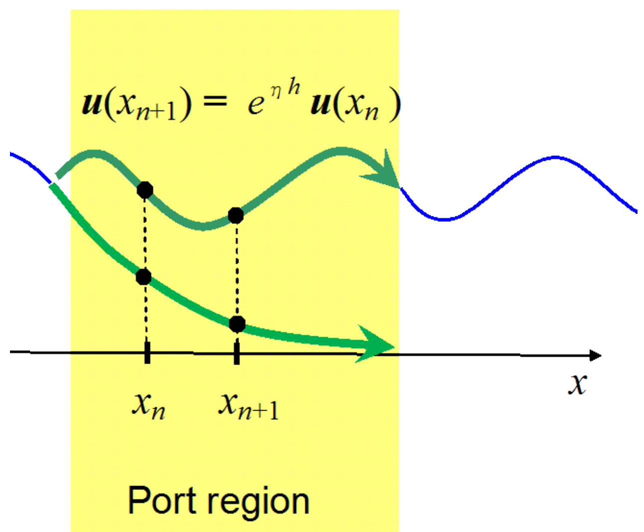
Eigen Problem and Stepping Matrix

$$\begin{bmatrix} c_{port} & b_{port} \\ O & I \end{bmatrix}\begin{bmatrix} \boldsymbol{u}(x_{n-1}) \\ \boldsymbol{u}(x_n) \end{bmatrix} = e^{\eta h}\begin{bmatrix} O & a_{port} \\ -I & O \end{bmatrix}\begin{bmatrix} \boldsymbol{u}(x_{n-1}) \\ \boldsymbol{u}(x_n) \end{bmatrix}$$

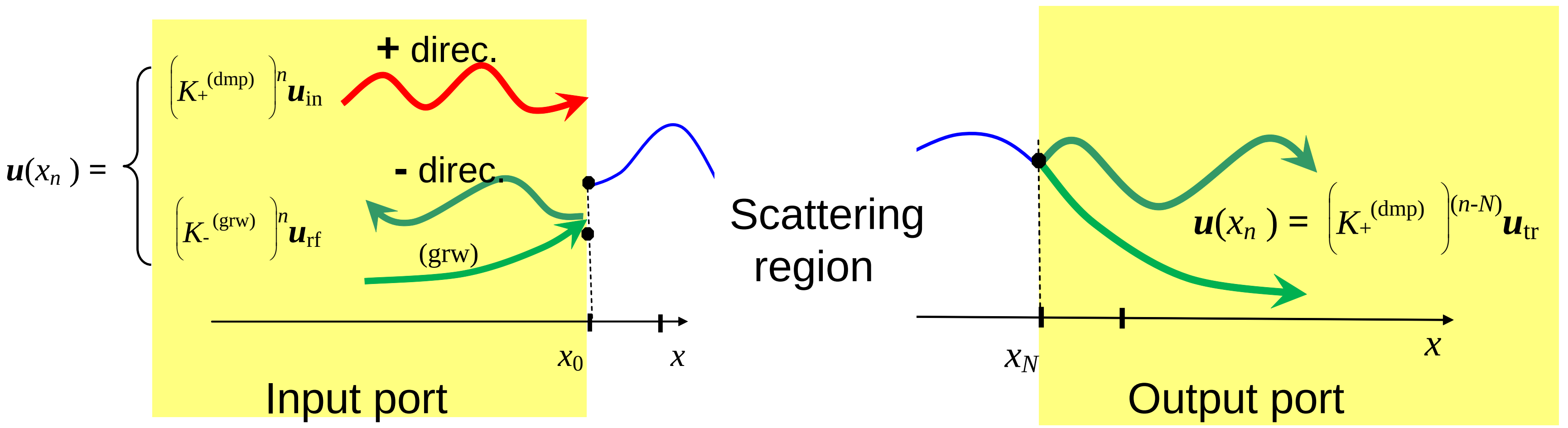
$$\eta = \begin{cases} \pm ik & \text{(propagation wave)} \\ \pm\gamma & \text{(dampen/grown wave)} \end{cases}$$

Stepping matrices

$$K_{\pm}^{(dmp/grw)} = V \begin{bmatrix} e^{\pm ikh} & 0 \\ 0 & e^{\pm\gamma h} \end{bmatrix} V^{-1}$$



Discretized wave expression in port regions



■Waves in Ports

$$\boldsymbol{u}(x_n) = \begin{cases} {K_+^{(\mathrm{dmp})}}^{\,n}\, \boldsymbol{u}_{\mathrm{in}} + {K_-^{(\mathrm{grw})}}^{\,n}\, \boldsymbol{u}_{\mathrm{rf}} & \text{input port} \\ {K_+^{(\mathrm{dmp})}}^{\,(n\text{-}N)}\, \boldsymbol{u}_{\mathrm{tr}} & \text{output port} \end{cases}$$

■Expressions for wave connections

$$\boldsymbol{u}(x_1) = K_+^{(\mathrm{dmp})}\, \boldsymbol{u}_{\mathrm{in}} + K_-^{(\mathrm{grw})}\, \boldsymbol{u}_{\mathrm{rf}} \qquad \text{at} \quad x = x_1$$

$$\boldsymbol{u}(x_{n+1}) = K_+^{(\mathrm{dmp})}\, \boldsymbol{u}(x_n) \qquad \text{at} \quad x_n > x_N$$

Generized stepping matrix in scattering region

- Stepping Matrix S_n

$$u(x_{n+1}) = S_n u(x_n)$$

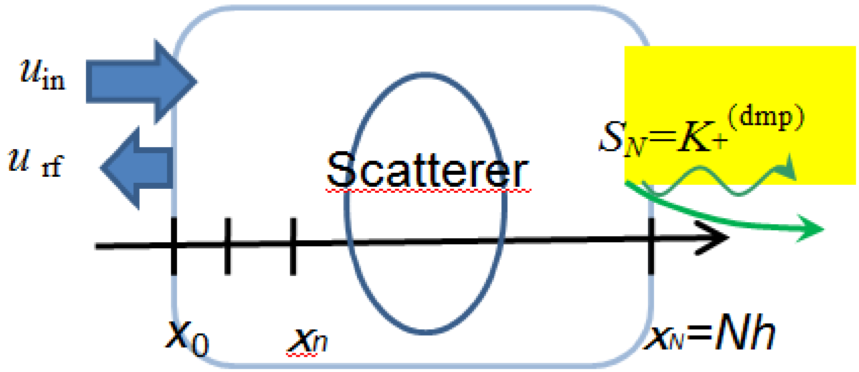
$$u(x_{n-1}) = (S_{n-1})^{-1} u(x_n)$$

$$c_n \mathbf{u}(x_{n-1}) + b_n \mathbf{u}(x_n) + a_n \mathbf{u}(x_{n+1}) = 0$$

$$c_n (S_{n-1})^{-1} + b_n + a_n S_n = 0$$

$$S_{n-1} = -(b_n + a_n S_n)^{-1} c_n$$

Recursion Relation



Simulation domain

- RTM-consistent PBC

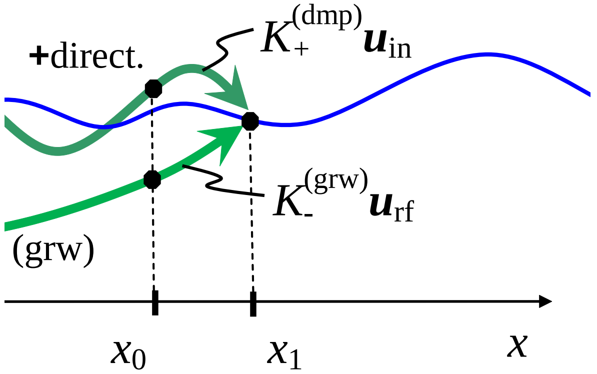
$$S_N = K_+^{(dmp)}$$

Absorbing BC in discretized system

Reflection and transmission coefficients by RTM

■Connection at x_1

$$K_+^{(dmp)}\,\boldsymbol{u}_{in} + K_-^{(grw)}\,\boldsymbol{u}_{rf} = S_0\,(\boldsymbol{u}_{in} + \boldsymbol{u}_{rf})$$



Reflection field

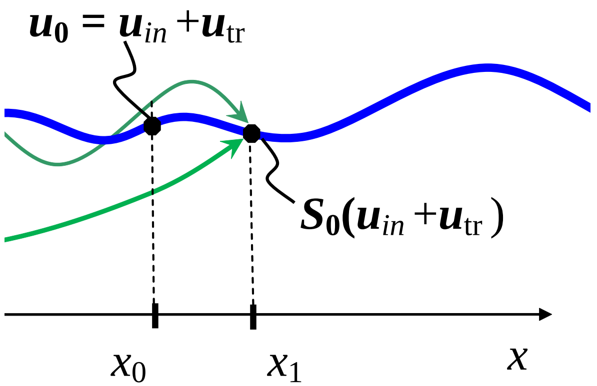
$$\boldsymbol{u}_{rf} = -(S_0 - K_-^{(grw)})^{-1}(S_0 - K_+^{(dmp)})\,\boldsymbol{u}_{in}$$

$r = \mathbf{e}_1 \cdot \boldsymbol{u}_{rf}$ Reflection coefficient

Transmission field

$$\boldsymbol{u}_{tr} = S_{N\text{-}1}\cdots S_2\,S_1 S_0\,(\boldsymbol{u}_{in} + \boldsymbol{u}_{rf})$$

$t = \mathbf{e}_1 \cdot \boldsymbol{u}_{tr}$ Transmission coefficient



$\mathbf{e}_1 = [1\ 0]^T$ Unit vecor

Why is the 2nd-order difference equation preferred?

- weak-form discretization scheme

  Extend the application scope of RTM

  4th-order diffrential eq. is discussed

- Stepping Matrices and recursion relation

  $K_{\pm}^{(\mathrm{dmp/grw})}$ in port regions

  Sn in scattering region

- RTM-consistent PBC

  An absorbing BC in discretized system

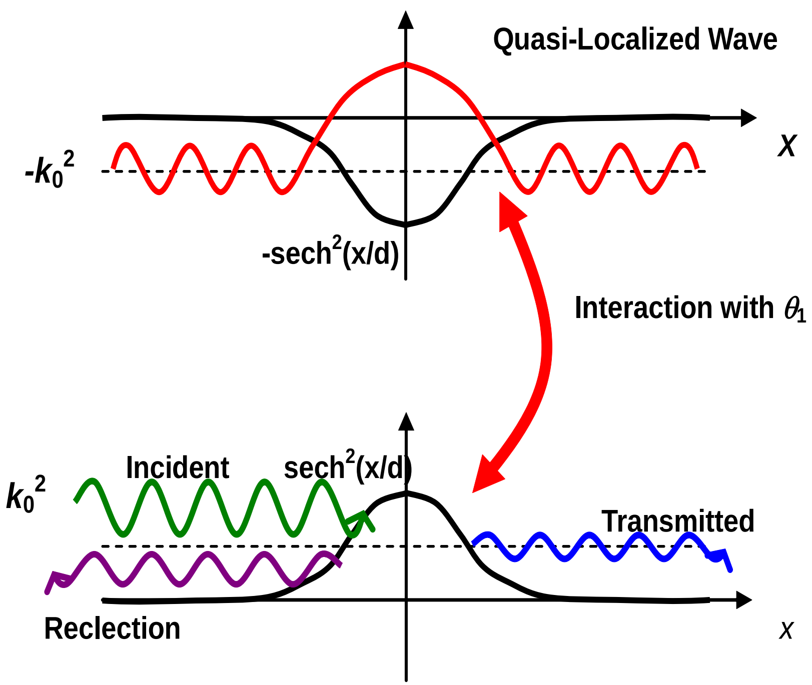
  Effective to extract quasi-localized waves

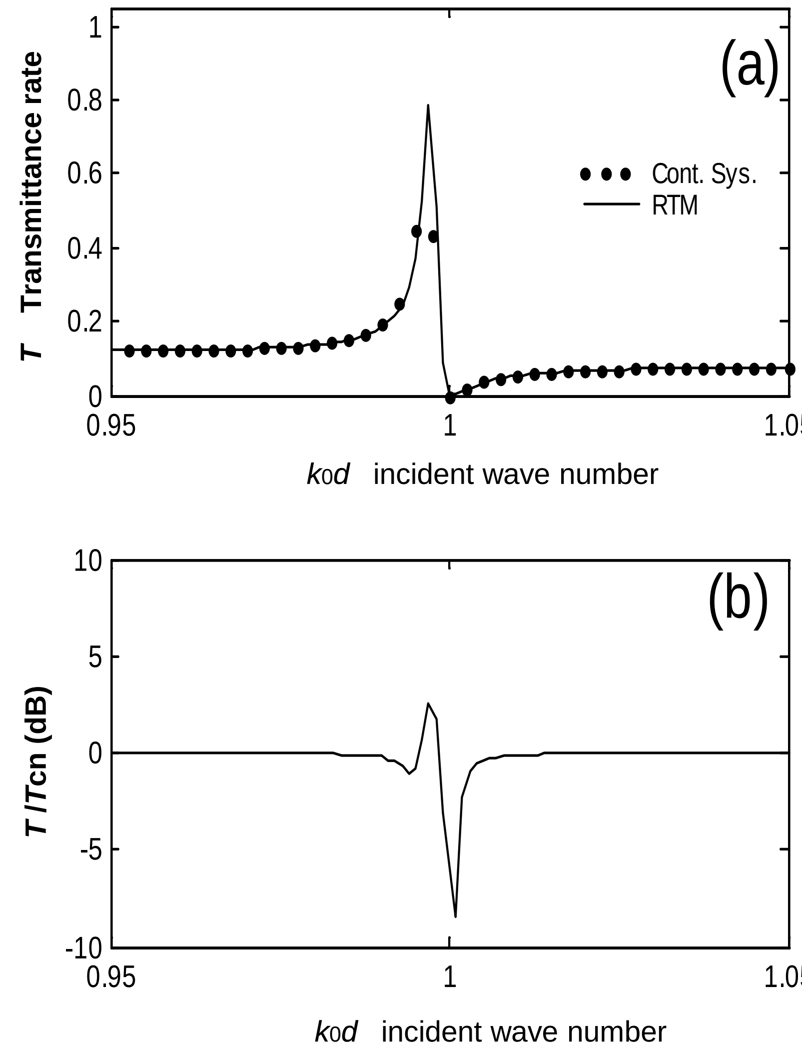
RTM applied to Fano resonance

- **Condition concerning to quasi-localized wave**

$$\sqrt{\frac{m(x)}{l(x)}} = k_0^2(1+r_b)\,\mathrm{sech}^2(x/d)$$

$$\sqrt{m(x)l(x)} = \begin{cases} e^{-2\theta_0} & x < 0 \\ e^{-2(\theta_0+\theta_1)} & x > 0 \end{cases}$$



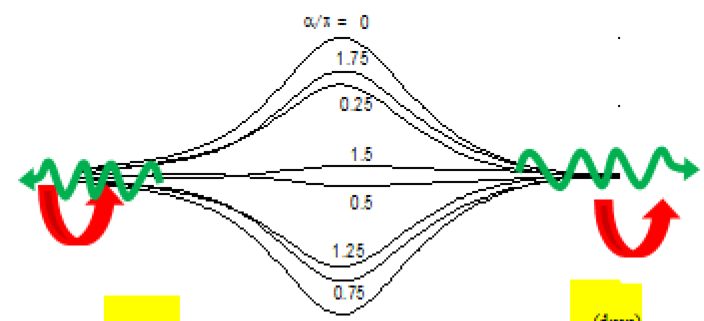


Extraction of quasi-localized wave (1/2)

■ RTM-consistent Port Boundary Condition

$$\boldsymbol{u}(x_{N+1}) = K_+^{(\mathrm{dmp})} \boldsymbol{u}(x_N)$$

$$\boldsymbol{u}(x_{-1}) = \left(K_-^{(\mathrm{grw})}\right)^{-1} \boldsymbol{u}(x_0)$$



$\boldsymbol{u}(x_0) = K_-^{(\mathrm{grw})}\ \boldsymbol{u}(x_{-1})$ $\qquad$ $\boldsymbol{u}(x_{N+1}) = K_+^{(\mathrm{dmp})} \boldsymbol{u}(x_N)$

■ Simultaneous eq. with all u(x_n)

$$c_0 \boldsymbol{u}(x_{-1}) + b_0 \boldsymbol{u}(x_0) + a_0 \boldsymbol{u}(x_1) = 0, \qquad (n=0)$$

$$c_n \boldsymbol{u}(x_{n+1}) + b_n \boldsymbol{u}(x_n) + a_n \boldsymbol{u}(x_{n-1}) = 0, \qquad (n=1,2,\ldots,N\text{-}1$$

$$c_N \boldsymbol{u}(x_{N-1}) + b_N \boldsymbol{u}(x_N) + a_N \boldsymbol{u}(x_{N+1}) = 0. \qquad (n=N)$$

■ Separation of coefficients with preliminary k_0^2 and its actual k_L^2

$$m(x) \Rightarrow (m(x) - k_0^2 l(x_0)) + k_L^2\, l(x_0)$$

$$c_n \Rightarrow c_n^{(1)} + k_L^2\, c_n^{(2)}, \qquad b_n \Rightarrow b_n^{(1)} + k_L^2\, b_n^{(2)}, \qquad a_n \Rightarrow a_n^{(1)} + k_L^2\, a_n^{(2)}.$$

Extraction of quasi-localized wave (2/2)

■ Actual $k_L^{\,2}$ as eigenvalue and wave shape as eigenvector

$$
\begin{bmatrix}
c_0^{(1)}\left(K_+^{(\mathrm{grw})}\right)^{-1}+b_0^{(1)} & a_0^{(1)} & 0 & \cdots & 0 & 0 \\
c_1^{(1)} & b_1^{(1)} & a_1^{(1)} & & & \\
0 & c_2^{(1)} & b_2^{(1)} & a_2^{(1)} & & \\
 & \ddots & & & & \\
0 & 0 & \cdots & c_1^{(1)} & b_N^{(1)}+a_N^{(1)} K_+^{(\mathrm{dmp})} &
\end{bmatrix}
\begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix}
= k_L^{\,2}
\begin{bmatrix}
c_0^{(2)}\left(K_+^{(\mathrm{grw})}\right)^{-1}+b_0^{(2)} & a_0^{(2)} & 0 & \cdots & 0 & 0 \\
c_1^{(2)} & b_1^{(2)} & a_1^{(2)} & & & \\
0 & c_2^{(2)} & b_2^{(2)} & a_2^{(2)} & & \\
 & \ddots & & & & \\
0 & 0 & \cdots & 2 & c_N^{(2)} & b_N^{(2)}+a_N^{(2)} K_+^{(\mathrm{dmp})}
\end{bmatrix}
\begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix}
$$

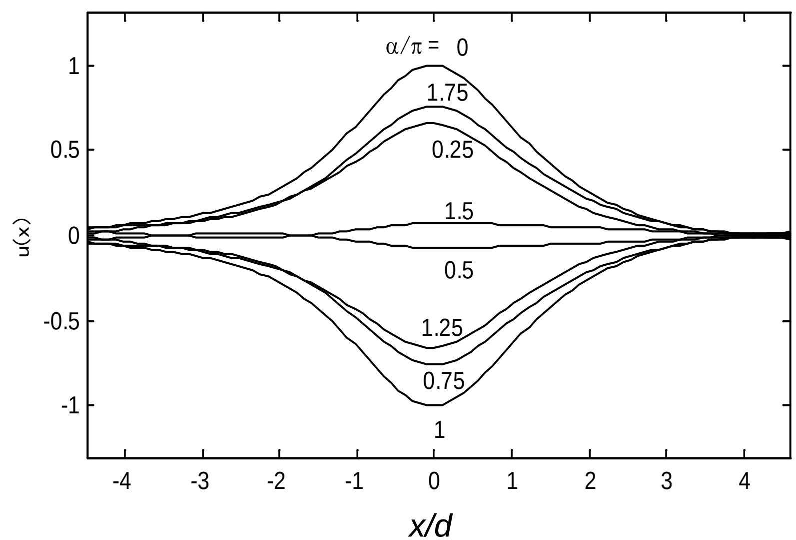
Consistency by iterative use

$$k_0 \;\longrightarrow\; K_+^{(\mathrm{dmp})},\; K_-^{(\mathrm{grw})} \longrightarrow\; k_L$$

Eleven times iterations

$$\left| \frac{k_L^{(p+1)} - k_L^{(p)}}{k_L^{(p)}} \right| < 10^{-5}$$

$k_L d = 0.99766 + i\, 0.00062$



S-matrix and its phase shift due to single pole *kL*

Unitarity of *S*-matrix

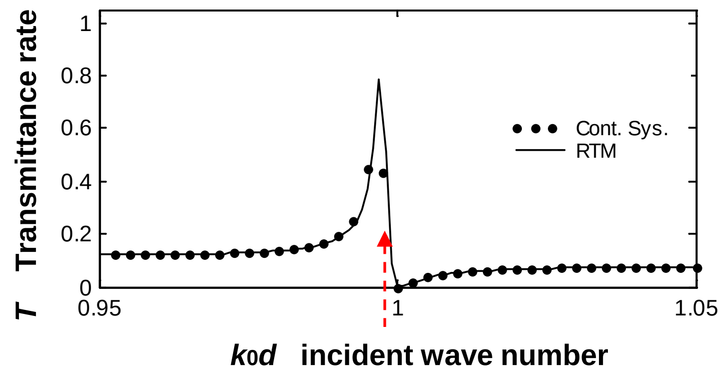
$$S(k) = \begin{bmatrix} r(k_0) & t'(k_0) \\ t(k_0) & r'(k_0) \end{bmatrix}$$

$$\det S(k_0) = e^{i2\,\delta\,(k_0)}$$

$$\approx e^{i2\,\delta_{bg}(k_0)} \frac{k_0 - k^*_L}{k_0 - k_L}$$

cf. scattering theory

$k_L d = 0.99766 + i\,0.00062$

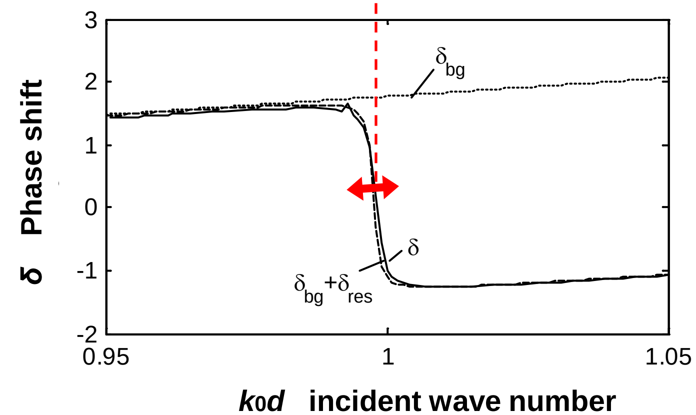


Decomposition of phase shift

$$\delta\,(k_0) \approx \delta_{bg}\,(k_0) + \delta_{res}\,(k_0)$$

$$\delta_{res}(k_0) = \frac{1}{2i} \log \frac{k_0 - k_L{}^*}{k_0 - k_L}$$

$$= \cot^{-1} \frac{k_0 - \mathrm{Re}[k_L]}{\mathrm{Im}[k_L]}$$



Wave equation for plate deformation(2D)

■ 4th-order differential equation

$$\Delta_0(D_0\Delta_0 u) + \Delta_1(D_1\Delta_1 u) + \Delta_2(D_2\Delta_2 u) - \omega^2 \rho b u = 0$$

Tensor Operators

$$\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\Delta_1 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \quad \Delta_2 = 2\frac{\partial^2}{\partial x^2 \partial y}$$

Plate Stifness

$$D_0(x,y) = \frac{Eb^3}{24(1-\nu)}$$

$$D_1(x,y) = D_2(x,y) = \frac{Eb^3}{24(1+\nu)}$$

■ Functional expression

$$F[w,u] = \int_L \{\Delta_0 w D_0 \Delta_0 + \Delta_1 w D_1 \Delta_1 u + \Delta_2 w D_2 \Delta_2 u - w\omega^2 \rho b u\} dS$$

$$\Rightarrow 0 \quad (\forall w)$$

BC for $w(x)$ $\quad w(x, y_n \pm h) = 0, \quad \frac{\partial}{\partial y} w(x, y_n \pm h) = 0$

■ 2nd-order difference equation

$$c_n \boldsymbol{U}(y_{n-1}) + b_n \boldsymbol{U}(y_n) + a_n \boldsymbol{U}(y_{n+1}) = \boldsymbol{0}$$

$$\boldsymbol{U}(y_n) = [\,\boldsymbol{u}_0(y_n)\ \boldsymbol{u}_1(y_n)\ \boldsymbol{u}_1(y_n)\boldsymbol{u}_2(y_n)\cdots\boldsymbol{u}_{Nx}(y_n)]^T,$$

$$\boldsymbol{u}_l(y_n) = [u(x_l, y_n)\ u_x(x_l, y_n)\ u_y(x_l, y_n)\ u_{xy}(x_l, y_n)]^T$$

Interpolation for discretization（2D, polynomial）

■ Cubic polynomial

x-direction

$$\boldsymbol{u}_l(y) = \begin{bmatrix} u(x_l\, y) \\ \dfrac{\partial}{\partial x} u(x_l\, y) \end{bmatrix}, \quad \xi = x - x_l,\ l = 0,1,2\ldots,N_x$$

$$u(x,y) = [1 \quad \xi \quad \xi^2 \ \xi^3]\, C \begin{bmatrix} \boldsymbol{u}_l(y) \\ \boldsymbol{u}_{l+1}(y) \end{bmatrix}$$

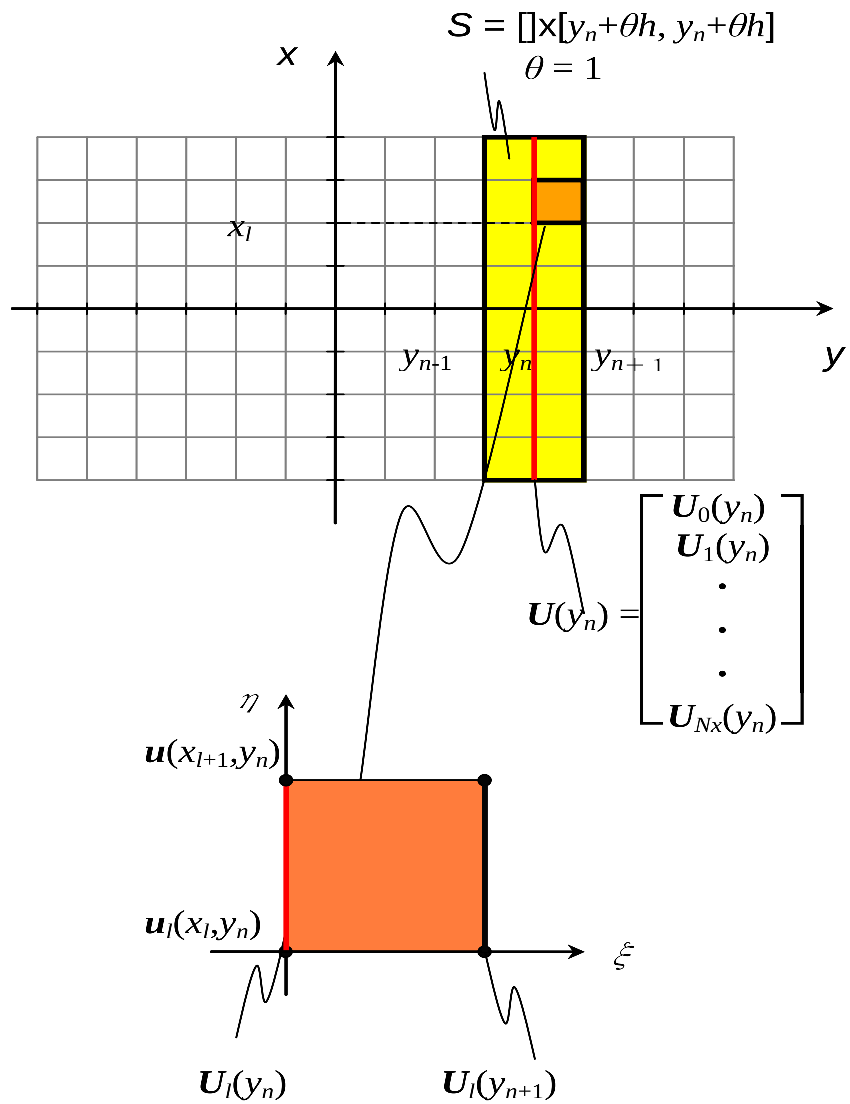
y-direction

$$\boldsymbol{U}_l(y_n) = \begin{bmatrix} \boldsymbol{u}_l(y_n) \\ \dfrac{\partial}{\partial y} \boldsymbol{u}_l(y_n) \end{bmatrix}, \quad \eta = y - y_n,\ n = 0,1,2\ldots,N_y$$

$$\boldsymbol{u}_l(y) = [1 \quad \eta \quad \eta^2 \ \eta^3]\, C \begin{bmatrix} \boldsymbol{U}_l(y_n) \\ \boldsymbol{U}_l(y_{n+1}) \end{bmatrix}$$

Serial vector

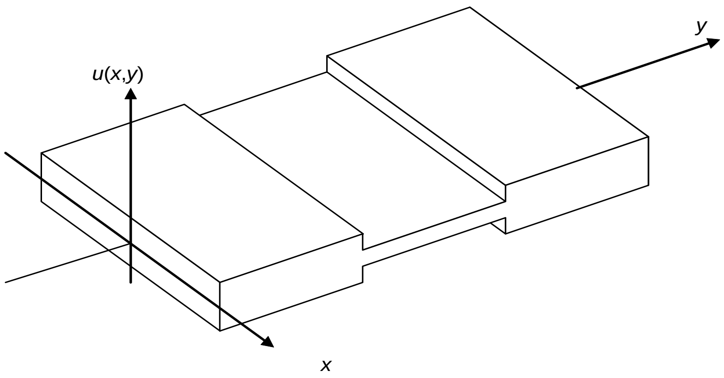
$$U(y_n) = [\ \boldsymbol{U}_0(y_n)\ \boldsymbol{U}_1(y_n)\ \boldsymbol{U}_2(y_n) \ldots.\ \boldsymbol{U}_{Nx}(y_n)]^T$$



$$a_n \boldsymbol{U}(y_{n+1}) + b_n \boldsymbol{U}(y_n) + c_n \boldsymbol{U}(y_{n-1}) = 0$$

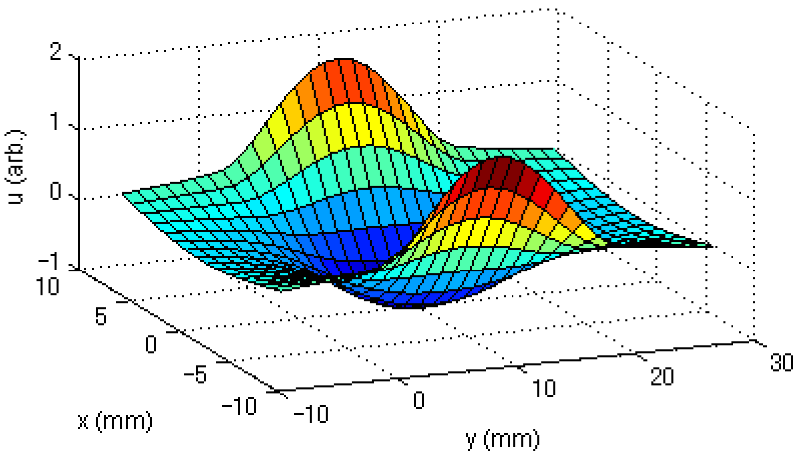
An extracted quasi-localized wave

Geometry of the system

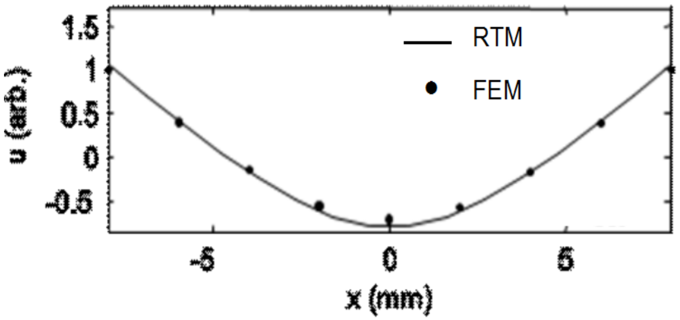


RTM with RTM-consistent PBC

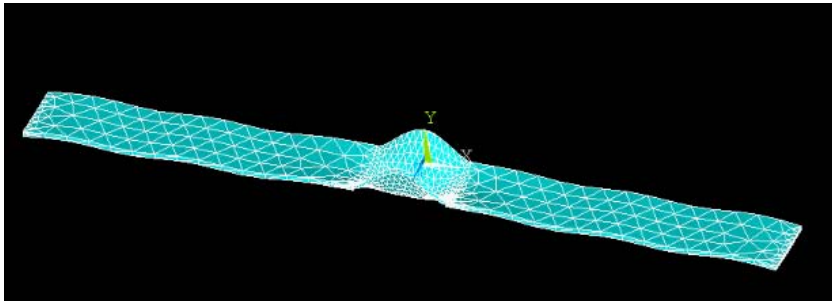
f_L = 15.4838 +i 0.02737 kHz



Comparison of mode shapes



FEM with free BC



Phase shift due to single pole f_L

Unitarity of S-matrix

$$S(k) = \begin{bmatrix} r(f) & t'(f) \\ t(f) & r'(f) \end{bmatrix}$$

$$\det S(f) = e^{i2\delta(f)}$$

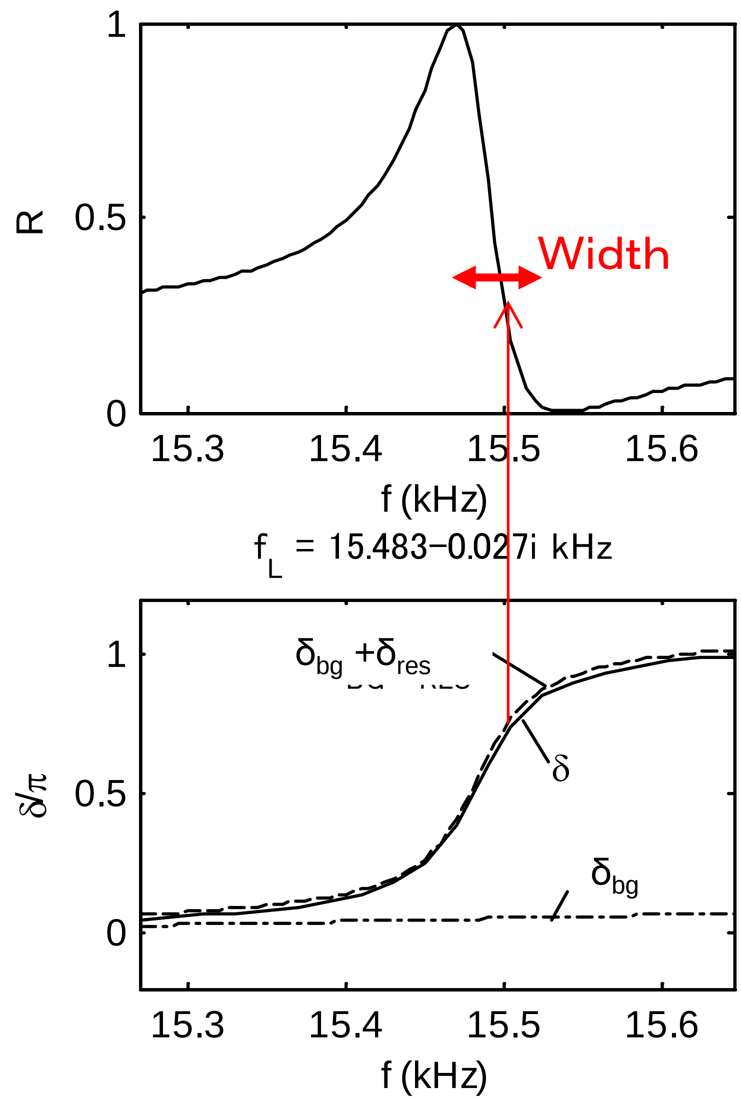
$$\approx e^{i2\delta_{bg}(f)} \frac{f - f_L^*}{f - f_L}$$

Phase shift decomposition

$$\delta(f) \approx \delta_{bg}(f) + \delta_{res}(f)$$

$$\delta_{res}(f) = \frac{1}{2i} \log \frac{f - f_L^*}{f - f_L}$$

$$= \cot^{-1} \frac{f - \mathrm{Re}[f_L]}{\mathrm{Im}[f_L]}$$



Quasi-localized wave as an eigenstate

- Consistency of k_0 with k_L and $K_{\pm}^{(\mathrm{dmp/grw})}$

- Validity of RTM-consistent PBC

- Single pole model for complex k_L

Concluding remarks

- Weak-form discretization to extend RTM scope
  Valid for any higher differential equations
  possessing a functional expression.

- RTM-consistent PBC
  Absorbing PBC in discretized system.
  Extraction of quasi-localized/localized waves.

- First discussion on 1D/2D Fano resonance being governed by 4^{th}-order differential equations.

References

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IEICE Transactions: Fundamentals, vol. E97-A no.8 pp. 1720-1727 (1 August 2014)

Lost of higher derivative in Neumann type BC

■For single mode

$$u = a_0 X_0$$

$$\frac{\partial u}{\partial y} = ik_0 a_0 X_0$$

$$\frac{\partial u}{\partial y} = ik_0 u$$

■For multi modes state

$$u = a_0 X_0 + a_1 X_1 + a_2 X_2$$

$$\frac{\partial u}{\partial y} = ik_0 a_0 X_0 + ik_1 a_1 X_1 + ik_2 a_2 X_2 ,$$

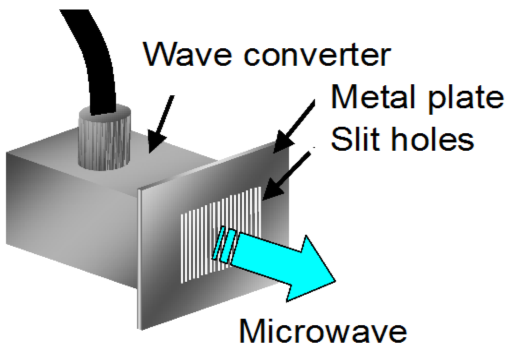
$$\frac{\partial^2 u}{\partial y^2} = (ik_0)^2 a_0 X_0 + (ik_1)^2 a_1 X_1 + (ik_2)^2 a_2 X_2,$$

$$\frac{\partial^3 u}{\partial y^3} = (ik_0)^3 a_0 X_0 + (ik_1)^3 a_1 X_1 + (ik_2)^3 a_2 X_2, .$$

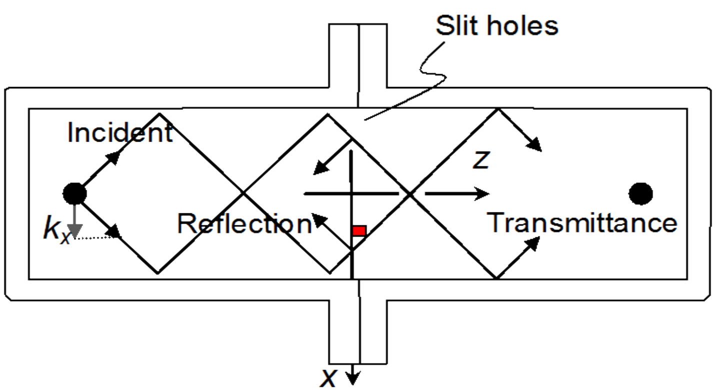
$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} X_0 & X_1 & X_2 \\ (ik_0)^2 X_0 & (ik_1)^2 X_1 & (ik_2)^2 X_2 \\ (ik_0)^3 X_0 & (ik_1)^3 X_1 & (ik_2)^3 X_2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^3 u}{\partial y^3} \end{bmatrix} .$$

$$\frac{\partial u}{\partial y} = [\, ik_0 X_0 \quad ik_1 X_1 \quad ik_2 X_2 \,] \begin{bmatrix} X_0 & X_1 & X_2 \\ (ik_0)^2 X_0 & (ik_1)^2 X_1 & (ik_2)^2 X_2 \\ (ik_0)^3 X_0 & (ik_1)^3 X_1 & (ik_2)^3 X_2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^3 u}{\partial y^3} \end{bmatrix} .$$

RTM confirmed by experiment



(a)



(b)